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STUDENT VERSION

Algal Blooms Threatening Lake Chapala

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STATEMENT

Background Reading

Lake Chapala, Mexico, hosts a diverse throng of fish and migratory birds as well as numerous fishing and farming communities [1]. The Lerma Rivers system, the ecological and hydrological core of central Mexico, drains into Lake Chapala before draining out to the Pacific Ocean. As such, problems in Lake Chapala are indicators of the health of the entire ecosystem.

The most visible ailment of Lake Chapala is dehydration due to evaporation, irrigation of it's headwaters to area farms, drainage by the Santiago River to the Pacific Ocean, and piping of potable water to Guadalajara's 4 million people. The lake has come close to drying up completely a couple times in the last century and currently sits half-full with an average depth of 4.5 meters (15 ft). In addition, the lake's once pristine waters are now muddy, with transparency rarely exceeding 30 centimeters (1 ft). As a combined result of the dehydration and increased deforestation and agriculture in the area, Lake Chapala has high levels of nitrogen, phosphorus and other sediments. Measured concentrations of these nutrients put the lake on the cusp of being hyper-eutrophic, having toxic levels of nutrients for the local food web [4]. Finally, the lake has been frequented by algal blooms, rapid increases in algae population, which have been irritations to the tourists and communities in the surrounding area [1].

Of the many factors that threaten Lake Chapala, we focus our attention on the algae, specifically referring to the free-floating phytoplankton of which the species Ankistrodesmus bibrainanus were studied in [2]. These algae are tiny drifting plants that grow and reproduce using photosynthesis. Over the past several hundred years of recorded observation of the lake, 1994 marked the first evidence of algal blooms in Lake Chapala [1]. Since then algal blooms have frequented the lake and the growth of seaweed and hyacinth have hampered the use of motorized fishing boats. Early studies found that the amount of nitrogen and the clay turbidity (cloudiness) of the water were the principal limiting factors to algae growth in Lake Chapala [2]. In fact, clay particles from erosion and agricultural runoff increase both the amount of nitrogen in the water and the turbidity of the water [4]. Algal blooms in the lake, though non-toxic so far, have tainted the smell and taste of potable water for the citizens of Guadalajara requiring further treatment of the water [1]. Currently no toxic strains of algae have been found in Lake Chapala. Yet, questions still linger concerning how the current strains of algae arrived and how the danger of toxic strains may be avoided [7].

A Citizens Report to the Commission for Environmental Cooperation in 2003 and further investigation purported that the Mexican government did not protect the water quality and ecology of the Lerma Rivers system and was lax in their enforcement of the environmental laws concerning the region [9]. The Mexican government later defended its compliance with environmental law in 2004 and its water management agency, the National Water Program, delineated its actions in restoring and preserving the water quality for the Lerma Rivers system [3]. Algae is just one factor in this environmental contention, but is highly visible due to its effect on the taste and smell of drinking water in Guadalajara. Thus it has been at the forefront in media reportings of the crisis of Lake Chapala [1].

Considering the government efforts to eliminate the algae in contrast with the media reportings of algal blooms of increasing freqency and magnitude [7], the resilience of the algae in Lake Chapala cannot be neglected. Let us consider the possibility that algal blooms are being forced periodically by an overwhelming influx of agricultural chemical residues: heavy metals, and dissolved solids, which provide a rich food base for the algae to rapidly grow [1].

Reading questions

1. What are some of the problems that algal blooms cause for the communities around Lake Chapala?
2. In addition to the agricultural chemical residues that we focus on in this activity, what other factors could contribute to algae growth?

Modeling Setup

To develop a resource-dependent model to describe the resilient ability of an algae population to bounce back after near-annihilation, start with a verbal model description as a guide.

$$\frac{du}{dt} = \text{Growth (Input) Function} - \text{Death (Output) Function} \tag{1}$$

The first growth function that should come to mind is the linear growth function,

$$\frac{du}{dt} = au, \ a > 0.$$

Unfortunately, the exponential solution to this model, $u(t) = e^{at}u(0)$, grows unbounded without any oscillations since the positive growth rate, a, is constant. Choose a growth function which is periodic in time to better match the information presented in the background reading.

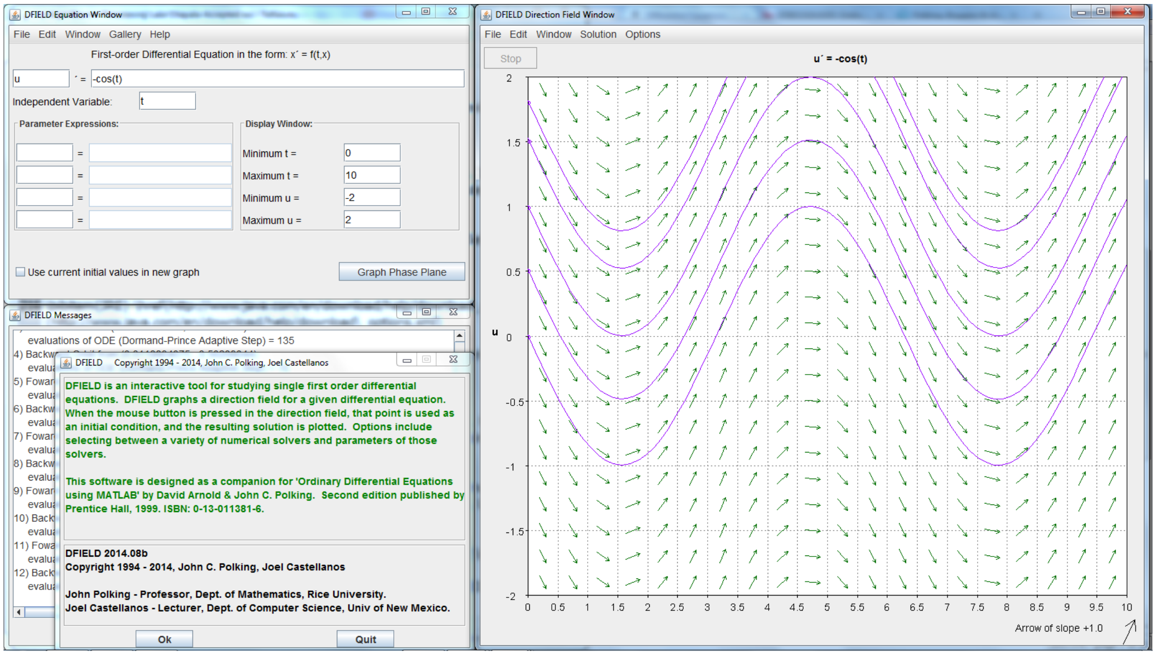

Figure 1. Display of DFIELD software plotting direction field of example model with several plotted solutions.

Using DFIELD Software

DFIELD is a powerful tool developed for teaching first order differential equations through visualizing and interactively analyzing directions fields [10]. The current version of DFIELD is a Java applet exported from MATLAB source code which can be run independent of MATLAB but needs a Java Runtime Environment [6] installed first. The current version, `DFIELD 2014.08b`, was updated in 2014 by Dr. Joel Castellanos at the University of New Mexico from code originally developed in 1994 by Dr. John Polking at Rice University, and is available as the file `dfield.jar`, for download at [11]. Documentation and original MATLAB source code is also available at the site.

Figure 1 shows an example display of DFIELD with an equation similar to model (1), but with an oscillating growth function:

$$\frac{dx}{dt} = -\cos(t)$$

Note that the change in growth (derivative) oscillates between -1 and $+1$. The solution, however, is not feasible because negative change in growth at zero population would create a negative population as demonstrated in Figure 1. The first activity shows how this first model version can be improved to be more realistic, both theoretically and visually.

Here is a quick tutorial for first users of DFIELD. Download and open the `dfield.jar` file. The copyright page gives a brief summary of what the DFIELD software can do; click "Ok". In the DFIELD equation window, enter your differential equation by typing the dependent variable u into the top left box, the derivative function in the box to the right, and the independent variable in the box below them both. Note that multiplied variables can be written sequentially with or without '*' between them, unlike the syntax of the underlying MATLAB code. Parameter values (e.g. a=1/2) need to be defined in the boxes in the lower left and the window size dimensions (min t = 0, max t = 1, min u = 0, max u = 1) go in the lower right. Note, the 't' and 'u' here refer to the independent and dependent variables. Clicking on "Graph Direction Field" will produce the direction field where you can click multiple times on the screen to see curves solved numerically using the clicked position as the initial position $(t_0, u(t_0))$.

ACTIVITIES

To form a population model from the verbal model (1), we seek simple functions for the derivative (amount of growth) so that the population stays feasible (nonnegative) and grows periodically in spite of consistent governmental interventions which reduce that growth. The following are a series of model versions improving incrementally towards the general model (2).

$$
\begin{aligned}
(V1)\ \frac{du}{dt} &= -\cos(t),\\
(V2)\ \frac{du}{dt} &= (1-\cos(t)),\\
(V3)\ \frac{du}{dt} &= (1-\cos(t))\,u,\\
(V4)\ \frac{du}{dt} &= (1-\cos(t))\,u-1,\\
(V5)\ \frac{du}{dt} &= (1-\cos(t))\,u-u,\\
\frac{du}{dt} &= a\,(1-\cos(bt))\,u-hu. \qquad (2)
\end{aligned}
$$

1. Use DFIELD to graph the five model versions V1-V5 and answer the following questions.
   (a) Version V1 begins with a cosine function to represent an oscillating growth based upon a periodic influx of chemicals. Does this oscillating amount of growth create an oscillating population as desired? Does this model begin with zero, positive, or negative growth?
   (b) Version V2 shifts the oscillation up. Why is it helpful to model the population growth by shifting up the oscillation by its amplitude?
   (c) Version V3 modifies the oscillation as a growth rate by writing it as a coefficient of the unknown function. Visually, how does the oscillating growth of version V2 differ from

the oscillating growth rate of version V3? Biologically, this is the difference between the amount of population growth (actual growth) and the percent population growth (rate of growth).

(d) Version V4 includes a constant output term to represent government intervention. Visually, how is the oscillating population affected? Biologically, this would represent harvesting of the algae. Is it feasible that equal numbers of algae were continuously removed?

(e) Version V5 modifies the output term to become a constant multiple of the unknown, making the amount removed proportional to the population at that time. Does the solution stay feasible (nonnegative) if the initial population is nonnegative?

(f) The general model (2) incorporates parameters a, b, h for flexibility in quantifying the desired behavior. What do each of the parameters represent and what are their units if time t is in days and population density u is in $\frac{g}{m^2}$? Note, a population density is more realistic than a traditional population count since algae are too numerous to count individually but can easily be measured by mass in a sample container with a fixed volume.

2. Choose values of a, b, h to fit the following data: agricultural pesticides are sprayed on nearby farms roughly twice a month (estimated every 15 days), maximum growth rate recorded was 25%, and subsequent maximum growth rates diminish slowly over time (estimated 0.5% per day).

3. Nondimensionalize the general model (2) by scaling to reduce the number of parameters in the model from three to two.

   Hint: One simple method is to scale everything by a common factor and make substitutions, such as turning $\frac{du}{dt} = bu(1 - au)$ into $\frac{adu}{bdt} = au(1 - au)$ or $\frac{dx}{d\tau} = x(1 - x)$, where $x = au, \tau = bt$. Note, it may be helpful to rename all variables and parameters in a common way. In this example, all parameters have been built into the new variables and thus will impact the solution behavior indirectly based upon bounds on x and τ.

4. Assuming $h \neq a$, find the general solution to your nondimensionalized model then substitute back in the original parameters a, b, h and apply the initial condition $u(0) = u_0$.

5. From the range of measurements of algae by [2], use an initial population density of $81 \frac{\mathrm{g}}{\mathrm{m}^2}$ with the parameter values previously computed to write out the specific solution and graph it. Assuming that population densities above $100 \frac{\mathrm{g}}{\mathrm{m}^2}$ trigger offensive odors in the drinking water of Guadalajara, estimate the length of each interval the algae population density exceeded this limit and the last time it did so (rounding to the nearest day for each).

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