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SIMIODE Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations

TEACHER VERSION
SALES MARKETING

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STATEMENT

In 1969 Frank M. Bass, an industrial administration professor at Purdue University published a seminal paper[1] in which he presented the development of what was to be called the *Bass diffusion model* which "consists of a simple differential equation that describes the process of how new products get adopted in a population."[4]

> The model presents a rationale of how current adopters and potential adopters of a new product interact. The basic premise of the model is that adopters can be classified as innovators or as imitators and the speed and timing of adoption depends on their degree of innovativeness and the degree of imitation among adopters. The Bass model has been widely used in forecasting, especially new products' sales forecasting and technology forecasting. Mathematically, the basic Bass diffusion is a Riccati [differential] equation with constant coefficients.[4]

We quote extensively from [1], inserting equation numbers for reference, so that the reader may see the definition, purpose, context, assumptions, and formulation of the model. Occasionally, we have inserted information in brackets [. . .] for clarity.

> The concern of this paper is the development of a theory of timing of initial purchase of new consumer products. The empirical aspects of the work presented here deal exclusively with consumer durables. The theory, however, is intended to apply to the growth of initial purchases of a broad range, of distinctive "new" generic classes of products. Thus, we draw a distinction between new classes of products as opposed to new brands or new models of older products.

$\vdots$

Sales grow to a peak and then level off at some magnitude lower than the peak. The stabilizing effect is accounted for by the relative growth of the replacement purchasing component of sales and the decline of the initial purchase component. We shall be concerned here only with the timing of initial purchase.

$\vdots$

In the discussion which follows an attempt will be made to outline the major ideas of the theory as they apply to the timing of adoption.

Some individuals decide to adopt an innovation independently of the decisions of other individuals in a social system. We shall refer to these individuals as *innovators*. We might ordinarily expect the first adopters to be innovators. In the literature, the following classes of adopters are specified: (1) Innovators; (2) Early Adopters; (3) Early Majority; (4) Late Majority; and (5) Laggards. This classification is based upon the timing of adoption by the various groups.

Apart from innovators, adopters are influenced in the timing of adoption by the pressures of the social system, the pressure increasing for later adopters with the number of previous adopters. In the mathematical formulation of the theory presented here we shall aggregate groups (2) through (5) above and define them as *imitators*. Imitators, unlike innovators, are influenced in the timing of adoption by the decisions of other members of the social system. Rogers [3] defines innovators, rather arbitrarily, as the first two and one-half percent of the adopters. Innovators are described as being venturesome and daring. They also interact with other innovators. When we say that they are not influenced in the timing of purchase by other members of the social system, we mean that the pressure for adoption, for this group, does not increase with the growth of the adoption process. In fact, quite the opposite may be true.

In applying the theory to the timing of initial purchase of a new consumer product, we formulate the following precise and basic assumption which, hopefully, characterizes the literary theory: The probability that an initial purchase will be made at T given that no purchase has yet been made is a linear function of the number of previous buyers. Thus,

$$P(T) = p + \frac{q}{m} Y(T)\,, \tag{1}$$

where p and $\frac{q}{m}$ are constants and $Y(T)$ is the number of previous buyers. Since $Y(0) = 0$, the constant p is the probability of an initial purchase at $T = 0$ and its magnitude reflects the importance of innovators in the social system. Since the parameters of the modal depend upon the scale used to measure time, it is possible to select a unit of measure for time such that p reflects the fraction of all adopters who are innovators in the sense in which Rogers[3] defines them. The product $\frac{q}{m}$ times $Y(T)$ reflects the pressures operating on imitators as the number of previous buyers increases.

In the section which follows, the basic assumption of the theory will be formulated in terms of a continuous model and a density function of time to initial purchase. We shall therefore refer to the linear probability element as a likelihood.

The following assumptions characterize the model:

a) Over the period of interest ("life of the product") there will be m initial purchases of the product. Since we are dealing with infrequently purchased products, the unit sales of the product will coincide with the number of initial purchases during that part of the time interval for which replacement sales are excluded. After replacement purchasing begins, sales will be composed of both initial purchases and replacement purchases. We shall restrict our interest in sales to that time interval for which replacement sales are excluded, although our interest in initial purchase will extend beyond this interval.

b) The likelihood of purchase at time T given that no purchase has yet been made is

$$\frac{f(T)}{1-F(T)} = P(T) = p + \frac{q}{m}Y(T) = p + qF(T), \tag{2}$$

where $f(T)$ is the likelihood of purchase at T and

$$F(T) = \int_0^T f(t)\,dt \qquad F(0) = 0. \tag{3}$$

Since $f(T)$ is the likelihood of purchase at T and m is the total number purchasing during the period for which the density function was constructed,

$$Y(T) = \int_0^T S(t)\,dt \; = m\int_0^T f(t)\,dt \; = mF(T) \tag{4}$$

is the total number purchasing in the $(0,T)$ interval. Therefore, sales at T [are given by]

$$S(T) = mf(T) = P(T)(mY(T)) = \left(p + \frac{q}{m}\int_0^T S(t)\,dt\right)\left(m - \int_0^T S(t)\,dt\right). \tag{5}$$

Expanding this product we have

$$S(T) = pm + (q-p)Y(T) - \frac{q}{m}(Y(T))^2\,. \tag{6}$$

The behavioral rationale for these assumptions are summarized:

a) Initial purchases of the product are made by both "innovators" and "imitators," the important distinction between an innovator and an imitator being the buying influence. Innovators are not influenced in the timing of their initial purchase by the number of people who have already bought the product, while imitators are influenced by the number of previous buyers. Imitators "learn," in some sense, from those who have already bought.

b) The importance of innovators will be greater at first but will diminish monotonically with time.

c) We shall refer to p as the *coefficient of innovation* and q as the *coefficient of imitation*.

Since [from (2)]

$$f(T) = (p + qF(T))(1 - F(T)) = p + (q - p)F(T) - q(F(T))^2\,, \qquad (7)$$

in order to find $F(T)$ we must first solve this nonlinear differential equation:

$$F'(T) = p + (q - p)F(T) - qF(T)^2 \quad F(0) = 0,$$

or more comfortably using t rather than T,

$$F'(t) = p + (q - p)F(t) - qF(t)^2 \quad F(0) = 0\,. \qquad (8)$$

This differential equation, (8), for $F(t)$, the sales at time t, is an example of a Riccati equation with constant coefficients.

Bass Paper Conclusions

In Bass's paper [1] the author offers the following conclusion which we share here.

> The growth model developed in this paper for the timing of initial purchase of new products is based upon an assumption that the probability of purchase at any time is related linearly to the number of previous buyers. There is a behavioral rational for this assumption. The model implies exponential growth of initial purchase to a peak and then exponential decay.[1, p. 226]

Finally Bass offers this wisdom,

> If the model developed in this paper does nothing else, it does demonstrate vividly the slowing down of growth rates as sales near the peak. In focusing upon the vital theoretical issues the model may serve to aid management in avoiding some of the more obviously absurd forecasts as have been made in the past.[1, p. 226]

Bass's paper [1] has been a defining work and was deemed so influential in marketing and management science research that is was reprinted in 2004 in *Management Science*[2] as one of the ten most frequently cited papers in the 50-year history of that journal.

ACTIVITIES

Activity 0

Defend the statement[1, p. 217], "The likelihood of purchase at time T given that no purchase has yet been made is

$$\frac{f(T)}{1-F(T)} = P(T) = p + \frac{q}{m}Y(T) = p + qF(T)."$$

That is, explain why does the left hand fraction express what is necessary to make the equality reasonable.

Activity 1

a) Show how you derive the differential equation (8) for sales, $F(t)$, from the terms defined by Bass and from the relationship, (7).

b) Solve the differential equation model, (8) for $F(t)$. Of course you will have parameters p, q, and m, as well as $F(0)$ as constants in your efforts. Although, often $F(0) = 0$ if we are over a time span that includes the initiation of sales.

Activity 2

Figure 1 depicts the sales figures for power lawnmowers in the period 1949-1961 as well as the superimposed model from Bass's analysis using regression methods.

We visually estimated the data in Table 1 from the polygonal data presentation in Figure 1 and converted year 1949 to $t = 0$ with consecutive years as $t = 1, 2, 3, \ldots$.

0	1	2	3	4	5	6	7	8	9	10	11	12
540	1090	1250	1190	1350	1380	2700	3200	3340	3490	4250	3780	3560

Table 1. Sales data for power lawnmowers from 1949 ($t = 0$) to 1961 ($t = 12$).

a) Using a sum of square error approach with knowledge of "initial" sales levels, $S(0) = 540$, in this case estimate the parameters p, q, and m, for the data in Table 1.

b) Plot your solution to (8) using the parameter values of p, q, and m which you ascertained in (a) and compare your model with the data. Certainly, plot your model over the data as Bass did in 1969.

c) Determine the peak sales (time and amount) from your model and compare that with what the data offers.

d) Determine when the sales are increasing most rapidly from your model (time and amount).

e) Compare your model result with that of Bass as depicted in Figure 1.

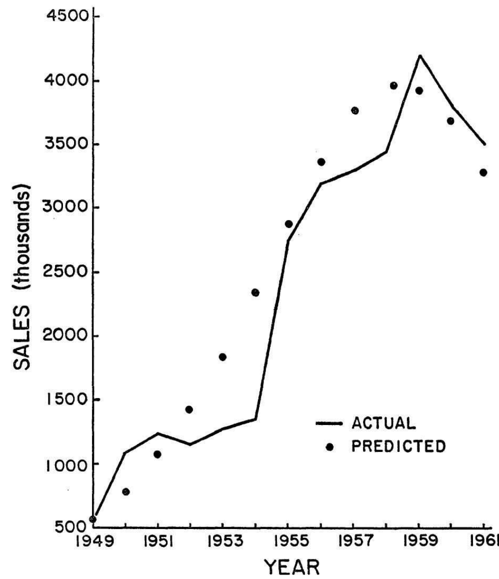

Fig. 7. Actual sales and sales predicted by model (power lawnmowers)

Figure 1. Sales data and Bass model fit for power lawnmowers. Source: [1, p. 223, Figure 7].

Activity 3

Use any one of the following data sets, reproduced in the Appendix, from Bass's original article with your model for sales, $S(t)$, to estimate the parameters, and address the issues in (a) through (d) of Activity 2 above.

APPENDIX - DATA

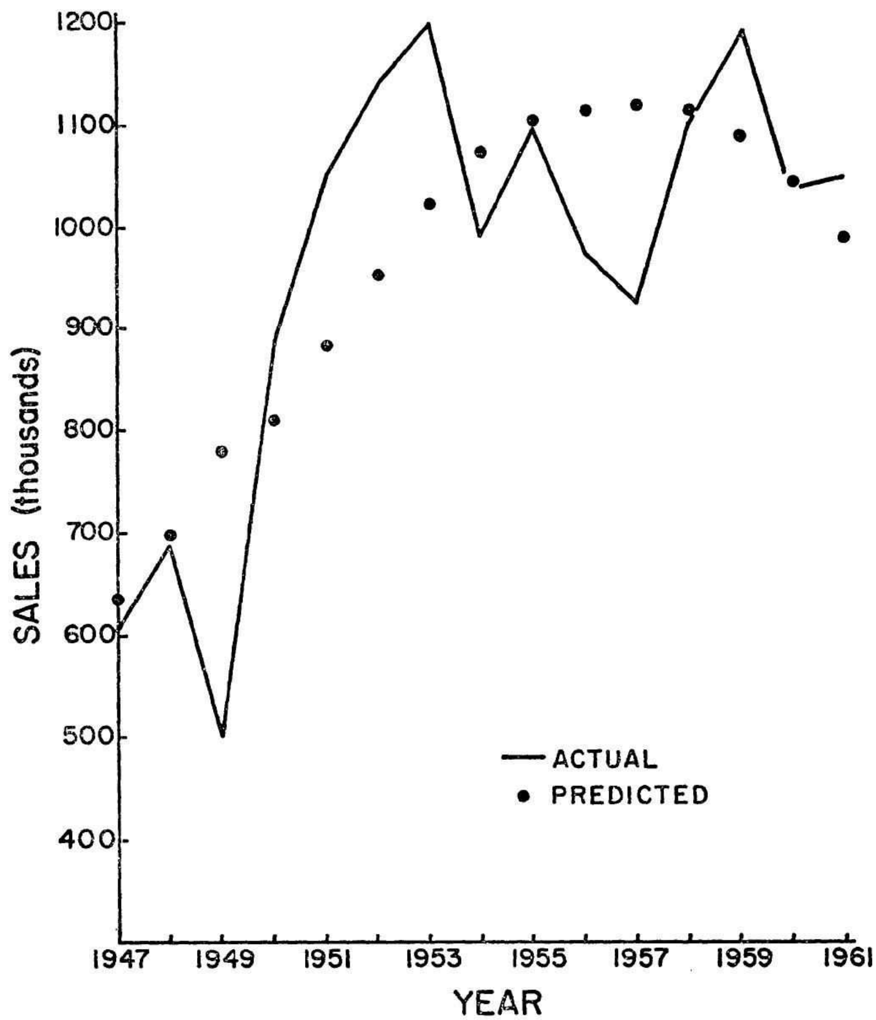

Fig. 5. Actual sales and sales predicted by regression equation (home freezers)

Figure 2. Sales data and Bass model fit for home freezers.
Source: [1, p. 220, Figure 5].

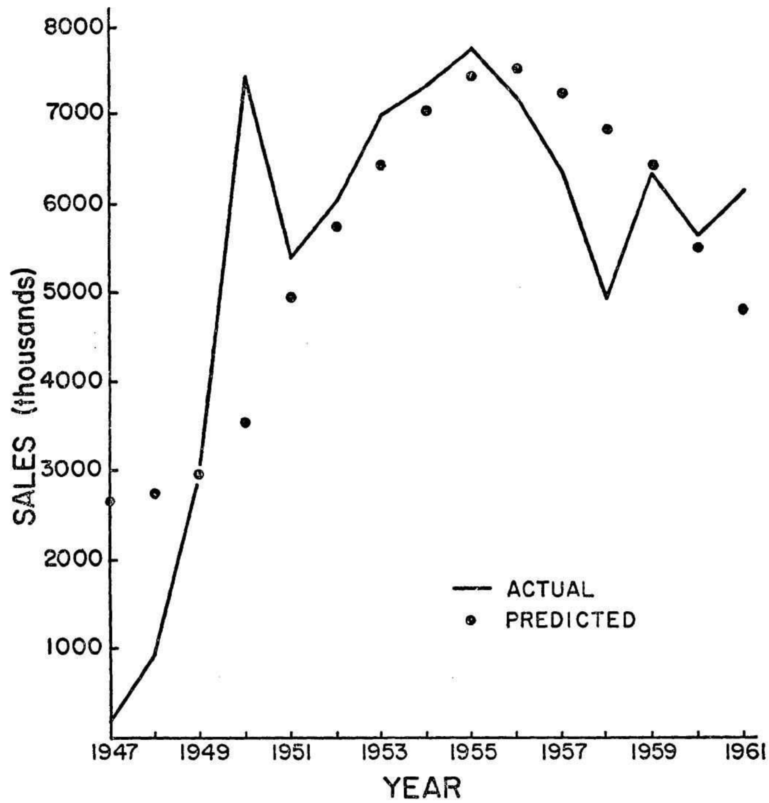

Fig. 6. Actual sales and sales predicted by regression equation (black & white television)

Figure 3. Sales data and Bass model fit for black and white televisions. Source: [1, p. 221, Figure 6].

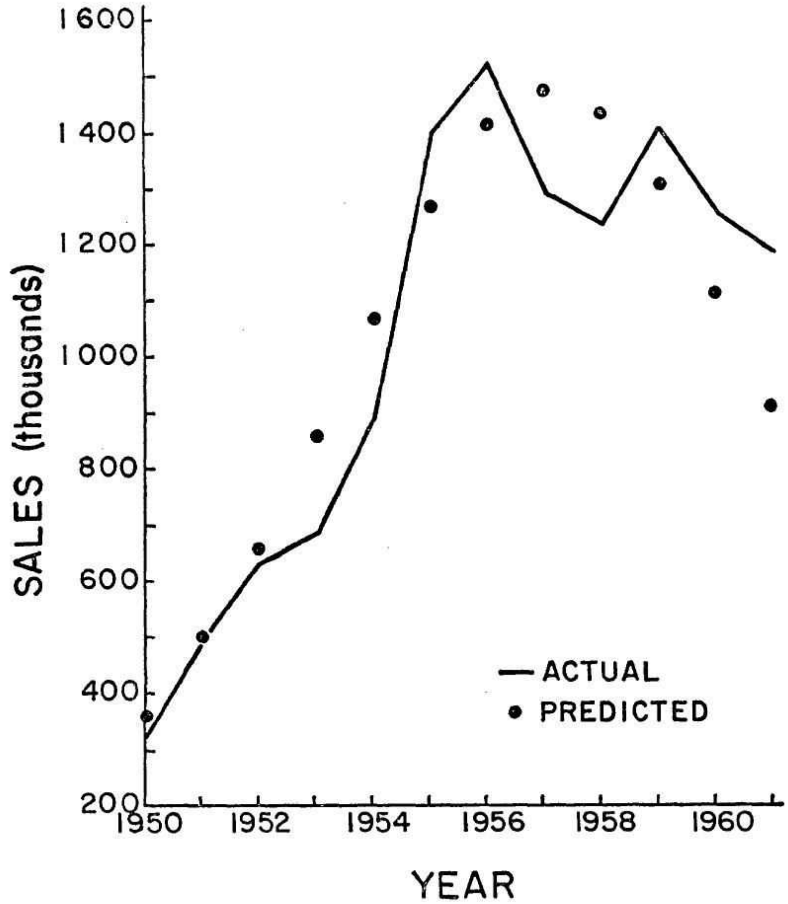

FIG. 8. Actual sales and sales predicted by model (clothes dryers)

Figure 4. Sales data and Bass model fit for clothes dryers.
Source: [1, p. 224, Figure 8].

REFERENCES

[1] Bass, F. M. 1969. A new product grown for model consumer durables. *Management Science*. 15(5): 215-227.

[2] Bass, F. M. 2004. A new product grown for model consumer durables. *Management Science*. Published online. 1 December 2004. http://pubsonline.informs.org/doi/abs/10.1287/mnsc.1040.0264. pp. 1825-1832. Accessed 13 November 2014.

[3] Rogers, E. M. 1962. *Diffusion of Innovation*. New York: The Free Press.

[4] Wikipedia contributors. 2014. Bass diffusion model. *Wikipedia, The Free Encyclopedia*. http://en.wikipedia.org/w/index.php?title=Bass_diffusion_model&oldid=621887389. Accessed 13 November 2014.